Solution to Assignment 13

No need to hand in any problem.

Section 9.3 no. 1(b)(d), 7, 8(a)(c), 14.

1. (b) Denote $x_n := \frac{1}{n+1} > 0$. Note $x_{n+1} - x_n = \frac{-1}{n(n+1)} < 0$, i.e. $\{x_n\}$ is decreasing and $\lim_n x_n = \lim_n \frac{1}{n+1} = 0$. By Alternating Series Test, this series is convergent. However,

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} x_n \right| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n+1} \geq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty \ .$$

Hence it converges conditionally.

(d) We have

$$\left(\frac{\log x}{x} \right)' = \frac{1 - \log x}{x^2}$$

which is negative for $x \geq e$. It follows that beginning from $n \geq 3$ $\{\log n/n\}_{n=3}^{\infty}$ is decreasing. By Alternating Series Test the series $\sum_{n=3}^{\infty} (-1)^{n+1} \log n/n$ is convergent, so is $\sum_{n=1}^{\infty} (-1)^{n+1} \log n/n$. On the other hand, it is clear that $\sum_{n=1} \log n/n = \infty$, so this series is conditionally convergent.

7. Let p, q be positive integer. Using the fact that $\log x/x^\alpha \to 0$ as $x \to \infty$ for any $\alpha > 0$, we have

$$\lim_{n\to\infty} \frac{(\log n)^p}{n^q} = 0.$$

Moreover, for large $x > 0$,

$$\frac{d}{dx}\left(\frac{\log x}{x^{\frac{p}{q}}} \right) = \frac{p - q \log x}{p x^{\frac{q}{p}+1}} < 0.$$

Therefore $(\log n)^p/n^q$ is decreasing for large n. By the Alternating Series Test, $\sum (-1)^n (\log n)^p/n^q$ converges.

8. (a) Denote $x_n := \frac{n^n}{(n+1)^{n+1}} = \frac{1}{(1+1/n)^n} \cdot \frac{1}{n+1}$. Since $n \mapsto \left(1 + \frac{1}{n}\right)^n$ is increasing (we learned this in 2050 when e was introduced),

$$x_{n+1} = \frac{1}{(1 + 1/(n+1))^{n+1}} \cdot \frac{1}{n+2} \leq \frac{1}{(1 + 1/n)^n} \cdot \frac{1}{n+1} = x_n \ .$$

Now

$$\lim x_n = \lim \frac{1}{(1 + 1/n)^n} \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0 \ .$$

By the Alternating Series Test, $\sum (-1)^n \frac{n^n}{(n+1)^{n+1}}$ converges.

Alternatively, observe that $\sum y_n = \sum (-1)^n/(n+1)$ is convergent and $x_n = n^n/(n+1)^n = (1 + 1/n)^n$ is increasing (to e). By Abel's test, this series is convergent.

(c) As $\lim \left| (-1)^n \frac{(n+1)^n}{n^n} \right| = e \neq 0$, this series cannot be convergent.

14. By Abel's Lemma, we have

$$\sum_{n+1}^{m} \frac{1}{k} a_k = \frac{1}{m} s_m - \frac{1}{n+1} s_n + \sum_{n+1}^{m} \left(\frac{1}{k} - \frac{1}{k+1} \right) s_k .$$

We have

$$\left(\frac{1}{k} - \frac{1}{k+1} \right) s_k \leq \frac{1}{k(k+1)} M k^r \leq M \frac{1}{k^{2-r}} .$$

As $r < 1, 2 - r > 1$ and $\sum_k k^{r-2} < \infty$. Hence $\sum_{n+1}^{m} k^{r-2} \to 0$ as $n, m \to \infty$. By comparison,

$$\sum_{n+1}^{m} \left(\frac{1}{k} - \frac{1}{k+1} \right) s_k \to 0 \ , \quad \text{as } n, m \to \infty \ .$$

On the other hand,

$$\left| \frac{1}{m} s_m - \frac{1}{n+1} s_n \right| \leq M \left(\frac{1}{m^{1-r}} + \frac{1}{n^{1-r}} \right) \to 0 \quad \text{as } n, m \to \infty \ .$$

We conclude that $\sum a_n/n$ is convergent.

Section 9.4 no. 5, 6(a)(c), 11, 12.

5. Let $L = \lim_{n\to\infty} |a_n|/|a_{n+1}|$. If $|\, x \,| < L$, $\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |\, x \,| < \frac{1}{L} \cdot L = 1$.
By the limit form of Ratio Test, $\sum a_n x^n$ converges absolutely if $|\, x \,| < L$.
If $|\, x \,| > L$, $\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \cdot |\, x \,| > \frac{1}{L} \cdot L = 1$. By the limit of Ratio Test, $\sum a_n x^n$ diverges if $|\, x \,| > L$. By Cauchy-Hadamard theorem, $R = L$.
If $L = 0$, then for $|\, x \,| > 0$, $\lim \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| \cdot |\, x \,| = |\, x \,| \lim \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
By ratio test, $\sum a_n x^n$ diverges if $|\, x \,| > 0$. By Cauchy-Hadamard theorem, $R = 0 = L$.
If $L = \infty$, then for $x \in \mathbb{R}$, $\lim \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| \cdot |\, x \,| = |\, x \,| \lim \left| \frac{a_{n+1}}{a_n} \right| = 0$.
By ratio test, $\sum a_n x^n$ converges if $|\, x \,| < \infty$. By Cauchy-Hadamard theorem, $R = \infty = L$.

Example: Consider the power series $1 + x^2 + x^4 + \cdots$. Here $a_{2n} = 1$ but $a_{2n+1} = 0$, so $\lim_{n\to\infty} |a_n/a_{n+1}|$ does not exist but $\rho = \limsup_{n\to\infty}(|a_n|^{1/n}) = 1$ and $R = 1$.

6. (a)

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0 \ .$$

Hence the radius of convergence is ∞.

(c)

$$\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n^n (n+1)!}{(n+1)^{n+1} n!} = \lim \left(1 + \frac{1}{n} \right)^{-n} = e^{-1} \ .$$

Hence the radius of convergence is e^{-1}.

11. Use Taylor expansion at point $x = 0$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} ,$$

for $|x| < r$ and $0 < |c| < |x|$. By assumption

$$\left| \frac{f^{(n+1)}(c)}{(n+1)!x^{(n+1)}} \right| \leq \frac{Br^{n+1}}{(n+1)!} .$$

Since $r^{n+1}/(n+1)! \to 0$ as $n \to \infty$, $\sum_{n=0}^{\infty} f^{(n)}(0)x^n/n!$ converges to $f(x)$ on $|x| < r$.

12. We did this exercise before. $f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$, set $t = 1/h$, we have

$$f'(0) = \lim_{t\to\infty} te^{t^2} ,$$

which clearly tends to 0. Assume that $f^{(k)}(0) = 0$, we want to show that $f^{(k+1)}(0) = 0$ We note that

$$f'(x) = 2x^{-3}e^{-1/x^2} \equiv P_3(1/x)e^{-1/x^2}$$

for $x \neq 0$, where P_3 is a polynomial with degree 3. We want to show that $f^{(n)}(x) = P_{3n}(1/x)e^{-1/x^2}$ for $x \neq 0$. Prove it by induction:
Assume that $f^{(k)}(x) = P_{3k}(1/x)e^{-1/x^2}$ for $x \neq 0$, then

$$f^{(k+1)}(x) = 2x^{-3}P_{3k}(1/x)e^{-1/x^2} + P_{3k-1}(1/x)e^{-1/x^2} \equiv P_{3(k+1)}(1/x)e^{-1/x^2} ,$$

for $x \neq 0$. Using this formula, it is easy to show that $f^{(n)}(0) = 0$ for all n. Hence this function is not given by its Taylor expansion about $x = 0$.

Supplementary Exercise

1. Let $f(x) = \sum_n a_n x^n$ whose radius of convergence is positive. Show that

$$a_n = \frac{f^{(n)}(0)}{n!} .$$

Solution. Let $R > 0$ be the radius of convergence of f. By Differentiation Theorem, f is smooth and termwise differentiation is valid on $(-R, R)$. Therefore, we have

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n x^{n-k} .$$

Taking $x = 0$ we conclude $f^{(k)}(0) = a_k k!$.

2. (a) Let $\sum_n a_n x^n$ be a power series which is convergent at non-zero x_0. Show that it converges uniformly on $[-r, r]$ for every $r \in [0, |x_0|)$.

(b) Deduce that if two power series $\sum_n a_n x^n$ and $\sum_n b_n x^n$ are equal on some $(-r, r), r > 0$. They are identical.

Solution. (a) Let R be the radius of convergence of this power series. If $|x_0| > R$, then $\sum_n a_n x^n$ is divergent according to Cauchy-Hadamard Theorem. Now, as it converges at x_0, we must have $|x_0| \le R$, so by C-H Theorem, the power series converges uniformly on $[-r, r]$ for every $r < |x_0|$.

(b) Let $\sum_n a_n x^n$ and $\sum_n b_n x^n$ be the two power series that are equal on $(-r, r), r > 0$. That means both series are uniformly convergent on $[-r_1, r_1]$ for every $r_1 < r$. Let $f(x) = \sum_n a_n x^n = \sum_n b_n x^n$. By (a), we must have $a_n = f^{(n)}(0)/n! = b_n$ for all n.